Energy stability bounds on convective heat transport: Numerical study

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The concept of nonlinear energy stability has recently been extended to deduce bounds on energy dissipation and transport in incompressible flows, even for turbulent flows. In this approach an effective stability condition on ''background'' flow or temperature profiles is derived, which when satisfied ensures that the profile produces a rigorous upper estimate to the bulk dissipation. Optimization of the test background profiles in search of the lowest upper bounds leads to nonlinear Euler-Lagrange equations for the extremal profile. In this paper, in the context of convective heat transport in the Boussinesq equations, we describe numerical solutions of the Euler-Lagrange equations for the optimal background temperature and present the numerical computation of the implied bounds. [S1063-651X(97)05706-1]

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The idea that notions of stability—usually reserved for the characterization of *static* stationary states—may be applied to *dynamic*, and even *turbulent*, phenomena has been proposed at various times and in various contexts. The modern concept of spontaneously self-imposed marginal stability was proposed in the context of thermal convection in the 1950s [1], later resulting in quantitative predictions for the bulk heat transport [2]. More recently, these sentiments have been given a rigorous formulation in the context of the dynamics of incompressible fluids. Building on a mathematical device invented by Hopf [3], and utilizing a decomposition referred to as the background field method [4,5], it has now become possible to formulate variational principles for upper bounds on the time averaged rate of heat transport where the key constraint on the background profiles over which the variation takes place is technically identical to a nonlinear energy stability [6] condition. The optimization problem produces nonlinear Euler-Lagrange equations for the ''marginally stable'' profile yielding the lowest upper bound. Interestingly, in some cases these Euler-Lagrange equations are of the same functional form as those found in Howard's theory of bounds on flow quantities in statistically stationary states [7], and with regard to the functional geometry of the constraints further connection has been established with Busse's multiple boundary layer theory [8] of Howard's bounds.

In this Brief Report we present the results of a numerical study of the solutions of the Euler-Lagrange equations for the optimal background profile. This study represents a practical implementation of the optimal methods developed in Ref. [5], and it illustrates some of the features that are expected in this kind of analysis. In particular, we observe that the optimal marginally stable profile may exist on different branches of solutions of the Euler-Lagrange equations, with the signal for the qualitative structural change being a change in ''stability'' of the solution.

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Consider an incompressible Newtonian fluid confined to the rectangular volume between rigid noslip isothermal plates at $z=0$ and 1. A vertical temperature gradient is imposed, so in the usual nondimensional units the fluid's velocity vector field $\mathbf{u}(\mathbf{x},t)=(u_1,u_2,u_3)$ and temperature field $T(\mathbf{x},t)$ satisfy the Boussinesq equations

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\sigma\Delta\mathbf{u}+\sigma\mathrm{Ra}\mathbf{k}T, \tag{1}$$

$$\nabla\cdot\mathbf{u}=\mathbf{0}, \tag{2}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\Delta T, \tag{3}$$

where $\sigma=\nu/\kappa$ is the Prandtl number (a ratio of material parameters, the kinematic viscosity ν and thermal diffusivity κ) and $\mathrm{Ra}=g\alpha\delta Th^3/\nu\kappa$ is the Rayleigh number (g is the acceleration of gravity, α is thermal expansion coefficient, δT is the temperature drop across the gap, and h is gap width), and $\mathbf{k}$ is the unit vector in the z direction. The boundary conditions are no-slip ($\mathbf{u}=\mathbf{0}$) on the $z=0$ and $z=1$ planes, $T=0$ on top, and $T=1$ on bottom, periodic in the horizontal directions with periods L_x and L_y, respectively.

The Nusselt number is the ratio of the (largest possible long time averaged) total heat transport to the purely conductive heat transport:

$$\mathrm{Nu}=1+\left\langle\frac{1}{L_xL_y}\int_0^{L_x}dx\int_0^{L_y}dy\int_0^1 dz\; u_3T\right\rangle. \tag{4}$$

The following theorem is proved in Ref. [5]: Let $\tau(z)$ be a function satisfying the imposed temperature boundary conditions, i.e., $\tau(0)=1$ and $\tau(1)=0$. Define $\psi(z)$ according to

$$\psi(z)=1+\frac{d\tau(z)}{dz}, \tag{5}$$

so that

$$\int_0^1 \psi(z)\,dz = 0. \tag{6}$$

Then

$$\mathrm{Nu} \leqslant 1 + \inf\left\{ \tfrac{1}{2}\int_0^1 \psi(z)^2 dz \right\}, \tag{7}$$

where the minimization is performed over mean zero functions $\psi(z)$ constrained by the ''stability'' condition (''spectral'' constraint) that

$$0 \leqslant H_\psi\{\mathbf{v},\theta\} = \int_0^{L_x} dx \int_0^{L_y} dy \int_0^1 dz \left\{ \frac{1}{2Ra} |\boldsymbol{\nabla} \mathbf{v}|^2 + (\psi(z)-2)\,\nu_3\theta + \frac{1}{2}|\boldsymbol{\nabla}\theta|^2 \right\} \tag{8}$$

for all divergence-free vector fields $\mathbf{v}(\mathbf{x})$ and temperature fluctuation fields θ satisfying homogeneous boundary conditions, i.e., vanishing on the rigid boundaries.

The condition in Eq. (8) is justifiably referred to as a ''stability'' constraint because it is the direct analogy of the sufficient condition for nonlinear energy stability that would apply to $\tau(z)$ if it *were* a profile corresponding to a static solution of the Boussinesq equations, the only difference being a factor of 4 rescaling of the Rayleigh number. The condition in Eq. (8) is also called a spectral constraint because it is equivalent to the non-negativity of the lowest (ground state) eigenvalue $\lambda^{(0)}$ of the self-adjoint problem

$$\lambda \nu_1 = -\sigma\Delta\nu_1 + \frac{\partial p}{\partial x}, \tag{9}$$

$$\lambda \nu_2 = -\sigma\Delta\nu_2 + \frac{\partial p}{\partial y}, \tag{10}$$

$$\lambda \nu_3 = -\sigma\Delta\nu_3 + \frac{\partial p}{\partial z} + \sigma\ \mathrm{Ra}(\psi(z)-2)\,\theta, \tag{11}$$

$$0 = \frac{\partial \nu_1}{\partial x} + \frac{\partial \nu_2}{\partial y} + \frac{\partial \nu_3}{\partial z}, \tag{12}$$

$$\lambda\theta = -\Delta\theta + (\psi(z)-2)\,\nu_3, \tag{13}$$

with boundary conditions $\mathbf{v}=\mathbf{0}$ and $\theta=0$ for $z=0$ and 1, and everything periodic in x and y. In the above, p is a Lagrange multiplier enforcing incompressibility, and the eigenvalue λ is the Lagrange multiplier enforcing the natural normalization for the eigenfunctions,

$$1 = \frac{1}{\sigma\ \mathrm{Ra}} \|\mathbf{v}\|_2^2 + \|\theta\|_2^2. \tag{14}$$

In Ref. [5] it was shown that for all Ra, the constrained set of test functions $\psi(z)$ is both nonempty and convex. Nonemptyness of the set was established by constructing a ''stable'' profile, and then elementary inequalities and asymptotics were used to produce explicit estimates culminating in the rigorous upper bound $\mathrm{Nu} \leqslant (\mathrm{Ra}/36)^{1/2} - 1$, uniform in the Prandtl number σ. Convexity ensures that there is a unique solution to the optimization problem. The optimal profile is just marginally ''stable,'' and the Euler-Lagrange equations are the ground-state problem Eqs. (9)–(13) closed by

$$\psi(z) = \alpha\left(\int_0^{L_x} dx \int_0^{L_y} dy\ \nu_3^{(0)}(x,y,z)\,\theta^{(0)}(x,y,z) - \int_0^1 dz' \int_0^{L_x} dx' \int_0^{L_y} dy'\, \nu_3^{(0)}(x',y',z') \times \theta^{(0)}(x',y',z') \right), \tag{15}$$

where α is the Lagrange multiplier used to enforce the marginal stability constraint, and the superscripts (0) indicate the ground-state eigenfunctions associated with ψ. Altogether, Eqs. (9)–(15) constitute a nonlocal nonlinear elliptic boundary value problem in which the Lagrange multiplier α is to be adjusted so that the lowest eigenvalue $\lambda^{(0)}=0$. The optimal temperature profile is subsequently reconstructed from the resulting eigenfunctions $\mathbf{v}^{(0)}$ and $\theta^{(0)}$.

Because the spectral problem is translation invariant in the horizontal directions, it may be separated via the Fourier transform. The general solution to the Euler-Lagrange equations may be a combination of any number of individual horizontal wave numbers, but for the case of a single wavenumber the problem may be reduced (see Ref. [5] for a detailed derivation) to

$$0 = k^{-2}(-D^2+k^2)^2 w + \sqrt{\mathrm{Ra}}(\psi(z)-2)\,\theta, \tag{16}$$

$$0 = (-D^2+k^2)\,\theta + \sqrt{\mathrm{Ra}}(\psi(z)-2)\,w, \tag{17}$$

where $D=\partial/\partial z$, the boundary conditions are $0=w(0)=Dw(0)=w(1)=Dw(1)=\theta(0)=\theta(1)$, and

$$\psi(z) = \alpha[w(z)\,\theta(z) - 1], \tag{18}$$

and α is to be adjusted so that the solution is normalized according to

$$1 = \int_0^1 w(z)\,\theta(z)\,dz. \tag{19}$$

The bound is then achieved by maximizing the upper estimate in Eq. (7) over the horizontal wave number k^2, and then checking that the resulting background profile is indeed ''stable.'' That is, once the candidate optimal profile ψ^{opt} and the associated wave number k^{opt} has been identified by solving Eqs. (16)–(19), it must be confirmed that the ''stability'' condition is satisfied. This condition is equivalent to the non-negativity for *all* values of k^2 of all the eigenvalues of the linear problem

$$\mu w = k^{-2}(-D^2+k^2)^2 w + \sqrt{\mathrm{Ra}}(\psi^{\mathrm{opt}}-2)\,\theta, \tag{20}$$

$$\mu\theta = (-D^2+k^2)\,\theta + \sqrt{\mathrm{Ra}}(\psi^{\mathrm{opt}}-2)\,w, \tag{21}$$

with boundary conditions $0=w(0)=Dw(0)=w(1)=Dw(1)=\theta(0)=\theta(1)$. The low-lying spectrum of this lin-

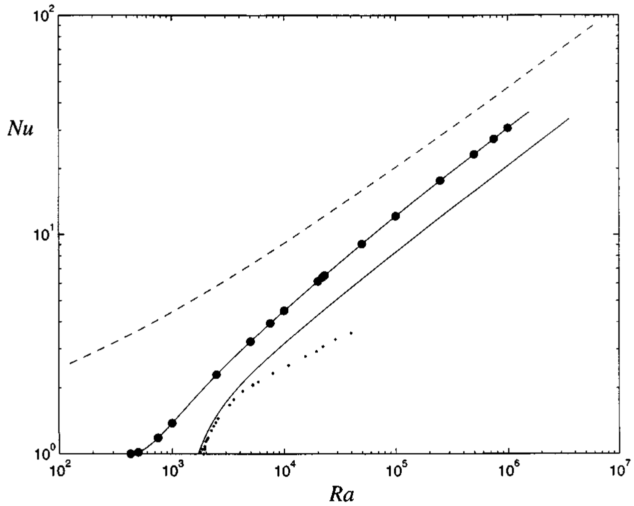

FIG. 1. Bounds on Nu as a function of Ra. The numerical background field result is the discrete data, with the solid line to guide the eye. The upper dashed line is the upper bound on the upper bound computed in Ref. [5], while the lower solid line is the bound for statistically stationary flows computed numerically from Howard's theory. The lower points are Rossby's experimental data for silicone oil.

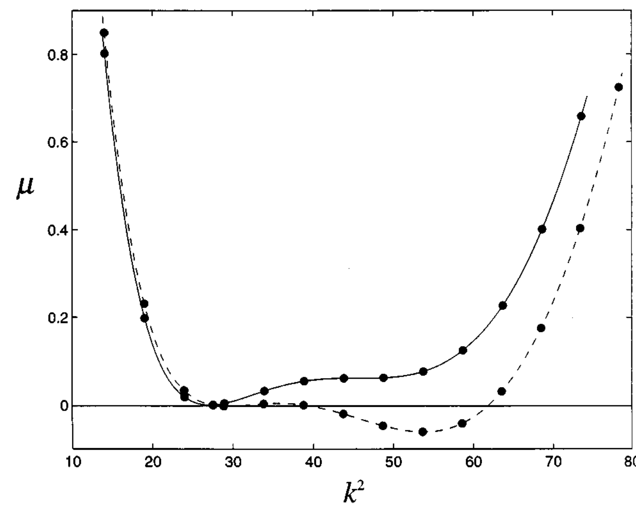

FIG. 2. Checking the ''stability'' of the optimal profile, the spectrum of the linear problem in Eqs. (20) and (21). For Ra=23 and 100 (solid line) the spectrum is non-negative, while for Ra $=23\,200$ (dashed line) there is a negative branch, indicating ''instability.''

ear operator is straightforward to calculate numerically once the optimal profile has been computed.

The problem in Eqs. (16)–(19) was solved numerically using using a relaxation method with five-point difference approximations for all the derivatives, second order for the fourth derivatives, and fourth order for the second derivatives. The leading truncation error terms were in the sixth derivative of the solution. At the boundaries, the solution was extended to two fictitious points where the extrapolation was chosen to satisfy a five-point difference approximation of the boundary conditions as well as the finite difference approximation of the equations at the boundary. This augmented the original boundary conditions with the compatibility conditions $D^2\theta(0)=0$ and $D^4w(0)=2k^2D^2w(0)$. The asymmetric extrapolation formula $\theta(-z)=-\theta(z)$ was used for θ near $z=0$. These boundary conditions introduce an $O(\Delta z)$ error in the discrete approximation which decays exponentially into the interior so that the global error was $O(\Delta z^2)$. Symmetry conditions were imposed at midinterval ($z=\frac{1}{2}$) and convergence was checked by comparing solutions with 41, 51, 61, and 71 discretization points on the interval $[0,\frac{1}{2}]$. The Lagrange multiplier was adjusted so that Eq. (19) was satisfied to within an error of 10^{-5}, evaluating the integral with a fourth-order cubic Hermite quadrature rule. Then, after finding the solution $\psi(z)$ for each value of k^2, $\int\psi^2$ was maximized over k^2.

The results of this procedure are the discrete data with the solid interpolation curve (to guide the eye) plotted in Fig. 1. The upper bound on the heat transport obtained in this way departs from the conduction value (Nu=1) at $\mathrm{Ra}=\mathrm{Ra}_c/4=427$ where $\mathrm{Ra}_c=1708$ is the critical value of the Rayleigh number where the conduction solution becomes unstable—unstable in the usual sense of both linear stability and nonlinear energy stability. The factor 4 rescaling is an artifact of particular choices made in the derivation of the variational bound in Ref. [5] and it has recently been shown in the context of shear flow that the variational principle may be quantitatively improved by making different choices in its formulation to bring the transition point right up to the energy stability boundary [9] (in this case the transition point for the bound may be brought right up to $\mathrm{Ra}_c=1708$) [10].

Also plotted in Fig. 1 for comparison are: (i) the rigorous upper estimate on the bound for the single-wave-number case as derived in Ref. [5], $\mathrm{Nu}\leq 1+0.257\,\mathrm{Ra}^{3/8}$ (top dashed line), (ii) the single-wave-number bound computed numerically [11,12] via Howard's method for statistically stationary flows (lower solid line), and (iii) Rossby's experimental heat transfer data [13] for silicon oil with Prandtl number $\sigma=200$ (lower discrete points). For large Ra, the upper bound computed by the optimal background field method in Fig. 1 scales $\sim\mathrm{Ra}^{3/8}$: an asymptotic analysis of the bound computed in Ref. [10] from Eqs. (16)–(19) yields the large Ra approximation $\approx 0.190\times\mathrm{Ra}^{3/8}$, about 26% below the rigorous upper bound on the bound $\sim 0.257\times\mathrm{Ra}^{3/8}$ obtained in Ref. [5] in very good agreement with the numerical results presented here, and just about 50% above the analogous asymptotic result for Howard's method.

The background profile result is not a valid upper bound on the heat transport unless the proposed background temperature profile is ''stable,'' however. This is determined by the spectrum of the eigenvalue problem in Eqs. (20) and (21), and in Fig. 2 we plot the spectrum of eigenvalues μ for two different proposed optimal profiles at two different values of Ra, as functions of the horizontal wave number. Up to Ra=23 100 (solid line), we observe that the spectrum is nonnegative with the lowest eigenvalue 0 precisely at the wave number corresponding to the ''marginally stable'' optimal mode, near $k^2\approx 28$ at Ra=23 100. However, for Ra $=23\,200$ (dashed line) we find that while the ''marginally stable'' optimal mode remains near $k^2\approx 28$, a set of modes at higher wave number around $k^2\approx 55$ has become ''unstable.'' Hence the single-wave-number solution of the Euler-Lagrange equations for the extremal background profile no longer yields the optimal bound for $\mathrm{Ra}\geq 23\,200$. As discussed in detail in Ref. [5], the true optimal solution transfers to a branch involving two wave numbers as with the analogous transition predicted by Busse for Howard's bounds.

The numerical results presented here give a quantitative representation of the rigorous results developed in Ref. [5]. In particular, they allow for quantitative evaluation and a comparison of the bounds derived by the background field method with Howard's theory for statistically stationary flows. We observe that the approaches give qualitatively similar results for the problem at hand (and also when applied to shear flow [14]) but show quantitative differences which, as previously noted, depend on details of the formulation of the variational bounds. Indeed, in Ref. [10] identical bounds are obtained (by approximate methods) from both the background field method and Howard's theory. We note that the background field method is applicable to more general problems that do not have the geometric or statistical symmetry necessary to formulate Howard's theory [15,16]. One open question with regard to the background field method for thermal convection is to produce an accurate numerical estimate of the scaling and corrections to scaling on the heat transport bounds in the asymptotic limit $\mathrm{Ra}\rightarrow\infty$. Another challenge is to realize the full power of the background field method by exploiting as-yet neglected *statistical regularity* [17] of turbulent flows and emerging ideas of *statistical stability* [18].

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